Chapter 25

Matrices do not necessarily form part of a first course in QM, but for Economics degrees and some others they can play a central role. The exercises from this chapter should provide the opportunity to practice the basic arithmetic of matrices and to apply them in a variety of situations.

1.

$$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 12 & 15 \end{bmatrix}$$

2.

$$4 \times [4 \quad 2 \quad 1] = [16 \quad 8 \quad 4]$$

3.

$$[4 \quad 2 \quad 1]\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = [50]$$

4.

$$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 7 \\ 8 & 10 \end{bmatrix} = \begin{bmatrix} 18 & 24 \\ 60 & 78 \end{bmatrix}$$

5.

$$\begin{bmatrix} 5 & 7 \\ 8 & 10 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 38 & 40 \\ 56 & 58 \end{bmatrix}$$

6. BC is not possible.

7.

$$\begin{bmatrix} 4 & 12 & 8 \\ 7 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 14 & 18 \\ 8 & 2 & 7 \\ 10 & 8 & 4 \end{bmatrix}$$

8.

$$[4 \quad 2 \quad 1] \times \begin{bmatrix} 4 & 12 & 8 \\ 7 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix} = [39 \quad 53 \quad 45]$$

9.

$$\begin{bmatrix} 4 & 12 & 8 \\ 7 & 2 & 5 \\ 9 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 4+12+8 & 8+0+56 & 40+24+8 \\ 7+2+5 & 14+0+35 & 70+4+5 \\ 9+1+3 & 18+0+21 & 90+2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 64 & 72 \\ 14 & 49 & 79 \\ 13 & 39 & 95 \end{bmatrix}$$

10.

$$\begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix} \times \begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 6+16+100 \\ 6+0+20 \\ 6+56+10 \end{bmatrix} = \begin{bmatrix} 122 \\ 26 \\ 72 \end{bmatrix}$$

11.

$$\frac{1}{10-4}\begin{bmatrix} 5 & -1 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 5/6 & -1/6 \\ -2/3 & 1/3 \end{bmatrix}$$

12.

$$|A| = 4 \times \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} - 7 \times \begin{vmatrix} 12 & 8 \\ 1 & 3 \end{vmatrix} + 9 \times \begin{vmatrix} 12 & 8 \\ 2 & 5 \end{vmatrix} = 4(6-5) - 7(36-8) + 9(60-16)$$
$$= 4 - 196 + 396 = 204$$

$$\mathit{CofactorsMatrix} = \begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 7 & 5 \\ 9 & 3 \end{vmatrix} & \begin{vmatrix} 7 & 2 \\ 9 & 1 \end{vmatrix} \\ -\begin{vmatrix} 12 & 8 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 4 & 8 \\ 9 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & 12 \\ 9 & 1 \end{vmatrix} \\ \begin{vmatrix} 12 & 8 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 8 \\ 7 & 5 \end{vmatrix} & \begin{vmatrix} 4 & 12 \\ 7 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 24 & -11 \\ -28 & -60 & 104 \\ 44 & 36 & -76 \end{bmatrix}$$

$$\mathit{Transposed} = \begin{bmatrix} 1 & -28 & 44 \\ 24 & -60 & 36 \\ -11 & 104 & -76 \end{bmatrix}$$

$$\mathit{Inverse} = \frac{1}{204} \times \begin{bmatrix} 1 & -28 & 44 \\ 24 & -60 & 36 \\ -11 & 104 & -76 \end{bmatrix}$$

13. AB = 50, so $(AB)^{-1} = 1/50$

14.

$$\begin{bmatrix} 6 \\ 8 \\ 10 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 6 \\ 32 & 16 & 8 \\ 40 & 20 & 10 \end{bmatrix}$$

15.

$$\frac{1}{58}\begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix}\begin{bmatrix} -14 & 68 & 4 \\ 1 & -9 & 8 \\ 7 & -5 & -2 \end{bmatrix} = \frac{1}{58}\begin{bmatrix} -14+2+70 & 68-18-50 & 4+16-20 \\ -14+0+14 & 68+0-10 & 4+0-4 \\ -14+7+7 & 68-63-5 & 4+56-2 \end{bmatrix}$$
$$= \frac{1}{58}\begin{bmatrix} 58 & 0 & 0 \\ 0 & 58 & 0 \\ 0 & 0 & 58 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

16. by definition GF = FG = I_3

17.

$$\begin{bmatrix} 10 & 5 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 & 10 \\ 1 & 0 & 2 \\ 1 & 7 & 1 \end{bmatrix} = \begin{bmatrix} 10+5+8 & 20+0+56 & 100+10+8 \end{bmatrix}$$
$$= \begin{bmatrix} 23 & 76 & 118 \end{bmatrix}$$

18.

$$\begin{bmatrix}6\\8\\10\end{bmatrix}\begin{bmatrix}10 & 5 & 8\end{bmatrix} = \begin{bmatrix}60 & 30 & 48\\80 & 40 & 64\\100 & 50 & 80\end{bmatrix}$$

19.

$$\begin{bmatrix}10 & 5 & 8\end{bmatrix}\begin{bmatrix}6\\8\\10\end{bmatrix} = [60 + 40 + 80] = [180]$$

20. we know from 17 that HF = [23 76 118], so

$$\begin{bmatrix}6\\8\\10\end{bmatrix}\begin{bmatrix}23 & 76 & 118\end{bmatrix} = \begin{bmatrix}138 & 456 & 708\\184 & 608 & 944\\230 & 760 & 1180\end{bmatrix}$$

21.

$$x = A^{-1}b$$

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{16-6}\begin{bmatrix}4 & -2\\-3 & 4\end{bmatrix}\begin{bmatrix}11\\9\end{bmatrix} = \frac{1}{10}\begin{bmatrix}44 & -18\\-33 & +36\end{bmatrix} = \begin{bmatrix}2.6\\0.3\end{bmatrix}$$

22.

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{-18-6}\begin{bmatrix}-3 & -3\\-2 & 6\end{bmatrix}\begin{bmatrix}18\\14\end{bmatrix} = \frac{1}{-24}\begin{bmatrix}-54 & -42\\-36 & +84\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$$

23.

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{10+6}\begin{bmatrix}2 & 3\\-2 & 5\end{bmatrix}\begin{bmatrix}26\\4\end{bmatrix} = \frac{1}{16}\begin{bmatrix}52 + 12\\-52 + 20\end{bmatrix} = \begin{bmatrix}4\\-2\end{bmatrix}$$

24.

$$\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{70-30}\begin{bmatrix}7 & -10\\-3 & 10\end{bmatrix}\begin{bmatrix}6\\11\end{bmatrix} = \frac{1}{40}\begin{bmatrix}42 & -110\\-18 & +110\end{bmatrix} = \begin{bmatrix}-1.7\\+2.3\end{bmatrix}$$

25. First we find the inverse of A

$$|A| = 4\begin{vmatrix}5 & 7\\6 & -2\end{vmatrix} - 2\begin{vmatrix}3 & 1\\6 & -2\end{vmatrix} + 5\begin{vmatrix}3 & 1\\5 & 7\end{vmatrix} = 4 \times -52 - 2 \times -12 + 5 \times 16 = -104$$

$$Cofactors = \begin{bmatrix}\begin{vmatrix}5 & 7\\6 & -2\end{vmatrix} & -\begin{vmatrix}2 & 7\\5 & -2\end{vmatrix} & \begin{vmatrix}2 & 5\\5 & 6\end{vmatrix}\\ -\begin{vmatrix}3 & 1\\6 & -2\end{vmatrix} & \begin{vmatrix}4 & 1\\5 & -2\end{vmatrix} & -\begin{vmatrix}4 & 3\\5 & 6\end{vmatrix}\\ \begin{vmatrix}3 & 1\\5 & 7\end{vmatrix} & -\begin{vmatrix}4 & 1\\2 & 7\end{vmatrix} & \begin{vmatrix}4 & 3\\2 & 5\end{vmatrix}\end{bmatrix} = \begin{bmatrix}-52 & 39 & -13\\12 & -13 & -9\\16 & -26 & 14\end{bmatrix}$$

$$A^{-1} = \frac{1}{-104}\begin{bmatrix}-52 & 12 & 16\\39 & -13 & -26\\-13 & -9 & 14\end{bmatrix}$$

Now use this to solve the equation:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{-104}\begin{bmatrix} -52 & 12 & 16 \\ 39 & -13 & -26 \\ -13 & -9 & 14 \end{bmatrix}\begin{bmatrix} 15 \\ 47 \\ 7 \end{bmatrix} = \frac{1}{38}\begin{bmatrix} -780 + 564 + 112 \\ 585 - 611 - 182 \\ -195 - 423 + 98 \end{bmatrix} = \frac{1}{-104}\begin{bmatrix} -104 \\ -208 \\ -520 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

26. First find the inverse of A

$$|A| = 10 \times \begin{vmatrix} -5 & 2 \\ -1 & 5 \end{vmatrix} - 20 \times \begin{vmatrix} 3 & 4 \\ -1 & 5 \end{vmatrix} + 25 \times \begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} = 10 \times -23 - 20 \times 19 + 25 \times 26 = 40$$

$$Cofactors = \begin{bmatrix} \begin{vmatrix} -5 & 2 \\ -1 & 5 \end{vmatrix} & -\begin{vmatrix} 20 & 2 \\ 25 & 5 \end{vmatrix} & \begin{vmatrix} 20 & -5 \\ 25 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 10 & 4 \\ 25 & 5 \end{vmatrix} & -\begin{vmatrix} 10 & 3 \\ 25 & -1 \end{vmatrix} \\ \begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} & -\begin{vmatrix} 10 & 4 \\ 20 & 2 \end{vmatrix} & \begin{vmatrix} 10 & 3 \\ 20 & -5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -23 & -50 & 105 \\ -19 & -50 & 85 \\ 26 & 60 & -110 \end{bmatrix}$$

$$A^{-1} = \frac{1}{40}\begin{bmatrix} -23 & -19 & 26 \\ -50 & -50 & 60 \\ 105 & 85 & -110 \end{bmatrix}$$

Now use this to solve the equation:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{40}\begin{bmatrix} -23 & -19 & 26 \\ -50 & -50 & 60 \\ 105 & 85 & -110 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \\ 16 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 4 \\ -40 \\ 100 \end{bmatrix} = \begin{bmatrix} 0.1 \\ -1 \\ 2.5 \end{bmatrix}$$

27. Using Cramer's rule gives:

$$|A| = \begin{vmatrix} 4 & 2 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 4 \times -5 - 1 \times 4 + 2 \times 3 = -18$$

$$|A_1| = \begin{vmatrix} 57 & 2 & -1 \\ -12 & -1 & 2 \\ 31 & 2 & 1 \end{vmatrix} = 57 \times -5 - (-12) \times 4 + 31 \times 3 = -144$$

$$|A_2| = \begin{vmatrix} 4 & 57 & -1 \\ 1 & -12 & 2 \\ 2 & 31 & 1 \end{vmatrix} = 4 \times -74 - 1 \times 88 + 2 \times 102 = -180$$

$$|A_3| = \begin{vmatrix} 4 & 2 & 57 \\ 1 & -1 & -12 \\ 2 & 2 & 31 \end{vmatrix} = 4 \times -7 - 1 \times -52 + 2 \times 33 = 90$$

$$x = \frac{-144}{-18} = 8; \; y = \frac{-180}{-18} = 10; \; z = \frac{90}{-18} = -5$$

28. Using Cramer's rule gives:

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 4 & -1 & 0 \end{vmatrix} = 2 \times 3 - 0 + 4 \times 2 = 14$$

$$|A_1| = \begin{vmatrix} 7 & 1 & 1 \\ 9 & 1 & 3 \\ 1 & -1 & 0 \end{vmatrix} = 7 \times 3 - 9 \times 1 + 1 \times 2 = 14$$

$$|A_2| = \begin{vmatrix} 2 & 7 & 1 \\ 0 & 9 & 3 \\ 4 & 1 & 0 \end{vmatrix} = 2 \times -3 - 0 + 4 \times 12 = 42$$

$$|A_3| = \begin{vmatrix} 2 & 1 & 7 \\ 0 & 1 & 9 \\ 4 & -1 & 1 \end{vmatrix} = 2 \times 10 - 0 + 4 \times 2 = 28$$

$$a = \frac{14}{14} = 1; \ b = \frac{42}{14} = 3; \ c = \frac{28}{14} = 2$$

29. Inverse of the matrix (from MINITAB) is:

$$\begin{bmatrix} -0.057143 & 0.428571 & -0.028571 & 0.285714 \\ 0.428571 & 0.285714 & -0.285714 & -0.142857 \\ -0.142857 & -0.428571 & 0.428571 & -0.285714 \\ 0.171429 & -0.285714 & 0.085714 & 0.142857 \end{bmatrix}$$

$$so \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -0.057143 & 0.428571 & -0.028571 & 0.285714 \\ 0.428571 & 0.285714 & -0.285714 & -0.142857 \\ -0.142857 & -0.428571 & 0.428571 & -0.285714 \\ 0.171429 & -0.285714 & 0.085714 & 0.142857 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 10 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \end{bmatrix}$$

30. Inverse of matrix (from MINITAB) is:

$$\begin{bmatrix} 0.09091 & 0.36364 & -0.27273 & 0.27273 \\ 0.18182 & -0.27273 & 0.45455 & -0.45455 \\ -1.00000 & 1.00000 & 0.00000 & 1.00000 \\ -0.72727 & 0.09091 & 0.18182 & 0.81818 \end{bmatrix}$$

$$so \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.09091 & 0.36364 & -0.27273 & 0.27273 \\ 0.18182 & -0.27273 & 0.45455 & -0.45455 \\ -1.00000 & 1.00000 & 0.00000 & 1.00000 \\ -0.72727 & 0.09091 & 0.18182 & 0.81818 \end{bmatrix} \begin{bmatrix} 29 \\ 7 \\ 20 \\ 22 \end{bmatrix}$$

$$= \begin{bmatrix} 5.72727 \\ 2.45455 \\ 0 \\ 1.18182 \end{bmatrix}$$

31.

(a)

$$A = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix}; \ X = \begin{bmatrix} 100 \\ 100 \end{bmatrix}; \ D = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$$

$$AX + D = \begin{bmatrix} 0.1 & 0.4 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} + \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \end{bmatrix} + \begin{bmatrix} 50 \\ 50 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix} = X$$

(b)

$$X = (I - A)^{-1} D$$

$$I - A = \begin{bmatrix} 0.9 & -0.4 \\ -0.3 & 0.8 \end{bmatrix}: \ so \ \ (I - A)^{-1} = \frac{1}{0.72 - 0.12} \begin{bmatrix} 0.8 & 0.4 \\ 0.3 & 0.9 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 8 & 4 \\ 3 & 9 \end{bmatrix}$$

$$X = \frac{1}{6} \begin{bmatrix} 8 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 220 \\ 170 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1760 + 680 \\ 660 + 850 \end{bmatrix} = \begin{bmatrix} 406.67 \\ 251.67 \end{bmatrix}$$

(c)

	Inputs to:			
Outputs from	A	B	Final Demand	Total Output
A	40.667	100.668	220	406.67
B	122.001	50.334	170	251.67
Other	244.002	100.668		
	406.67	251.67		

32.

(a)

$$A = \begin{bmatrix} 0.05 & 0.17 \\ 0.2 & 0.24 \end{bmatrix}; \ X = \begin{bmatrix} 100 \\ 500 \end{bmatrix}; \ D = \begin{bmatrix} 10 \\ 360 \end{bmatrix}$$

$$AX + D = \begin{bmatrix} 0.05 & 0.17 \\ 0.2 & 0.24 \end{bmatrix} \begin{bmatrix} 100 \\ 500 \end{bmatrix} + \begin{bmatrix} 10 \\ 360 \end{bmatrix} = \begin{bmatrix} 90 \\ 140 \end{bmatrix} + \begin{bmatrix} 10 \\ 360 \end{bmatrix} = \begin{bmatrix} 100 \\ 500 \end{bmatrix} = X$$

(b)

$$X = (I - A)^{-1} D$$

$$I - A = \begin{bmatrix} 0.95 & -0.17 \\ -0.2 & 0.76 \end{bmatrix}: \ so \ \ (I - A)^{-1} = \frac{1}{0.722 - 0.034} \begin{bmatrix} 0.76 & 0.17 \\ 0.2 & 0.95 \end{bmatrix} = \frac{1}{688} \begin{bmatrix} 760 & 170 \\ 200 & 950 \end{bmatrix}$$

$$X = \frac{1}{688} \begin{bmatrix} 760 & 170 \\ 200 & 950 \end{bmatrix} \begin{bmatrix} 50 \\ 500 \end{bmatrix} = \frac{1}{688} \begin{bmatrix} 38000 + \ 85000 \\ 10000 + 475000 \end{bmatrix} = \begin{bmatrix} 178.77906 \\ 704.94186 \end{bmatrix}$$

33.

(a)

$$A = \begin{bmatrix} 0.2 & 0.05 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.4 \end{bmatrix}$$

(b)

$$(I - A) = \begin{bmatrix} 0.8 & -0.05 & -0.1 \\ -0.2 & 0.8 & -0.2 \\ -0.2 & -0.1 & 0.6 \end{bmatrix}$$

(c)

$$|(I - A)| = 0.8 \times 0.46 - -0.2 \times 0.04 + -0.2 \times 0.09 = 0.342$$

$$Cofactors = \begin{bmatrix} \begin{vmatrix} 0.8 & -0.2 \\ -0.1 & 0.6 \end{vmatrix} & -\begin{vmatrix} -0.2 & -0.2 \\ -0.2 & 0.6 \end{vmatrix} & \begin{vmatrix} -0.2 & 0.8 \\ -0.2 & -0.1 \end{vmatrix} \\ -\begin{vmatrix} -0.05 & -0.1 \\ -0.1 & 0.6 \end{vmatrix} & \begin{vmatrix} 0.8 & -0.1 \\ -0.2 & 0.6 \end{vmatrix} & -\begin{vmatrix} 0.8 & -0.05 \\ -0.2 & -0.1 \end{vmatrix} \\ \begin{vmatrix} -0.05 & -0.1 \\ 0.8 & -0.2 \end{vmatrix} & -\begin{vmatrix} 0.8 & -0.1 \\ -0.2 & -0.2 \end{vmatrix} & \begin{vmatrix} 0.8 & -0.05 \\ -0.2 & 0.8 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0.46 & 0.16 & 0.18 \\ 0.04 & 0.46 & 0.09 \\ 0.09 & 0.18 & 0.63 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{1000}{342}\begin{bmatrix} 0.46 & 0.04 & 0.09 \\ 0.16 & 0.46 & 0.18 \\ 0.18 & 0.09 & 0.63 \end{bmatrix} = \frac{10}{342}\begin{bmatrix} 46 & 4 & 9 \\ 16 & 46 & 18 \\ 18 & 9 & 63 \end{bmatrix}$$

(d) $X = (I - A)^{-1} D$

$$X = \frac{10}{342}\begin{bmatrix} 46 & 4 & 9 \\ 16 & 46 & 18 \\ 18 & 9 & 63 \end{bmatrix}\begin{bmatrix} 40 \\ 80 \\ 140 \end{bmatrix} = \frac{10}{342}\begin{bmatrix} 1840 + 320 + 1260 \\ 640 + 3680 + 2520 \\ 720 + 720 + 8820 \end{bmatrix} = \frac{10}{342}\begin{bmatrix} 3420 \\ 6840 \\ 10260 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

(e) New output requirements:

$$X = \frac{10}{342}\begin{bmatrix} 46 & 4 & 9 \\ 16 & 46 & 18 \\ 18 & 9 & 63 \end{bmatrix}\begin{bmatrix} 342 \\ 684 \\ 1026 \end{bmatrix} = 10\begin{bmatrix} 46 & 4 & 9 \\ 16 & 46 & 18 \\ 18 & 9 & 63 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 10\begin{bmatrix} 46 + 8 + 27 \\ 16 + 92 + 54 \\ 18 + 18 + 189 \end{bmatrix} = \begin{bmatrix} 810 \\ 1620 \\ 2250 \end{bmatrix}$$

34.

$$A = \begin{bmatrix} 0.05 & 0.025 & 0.01 \\ 0.05 & 0.05 & 0.02 \\ 0.5 & 0.75 & 0.07 \end{bmatrix}$$

$$(I - A) = \begin{bmatrix} 0.95 & -0.025 & -0.01 \\ -0.05 & 0.95 & -0.02 \\ -0.5 & -0.75 & 0.93 \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 1.06104 & 0.03757 & 0.01222 \\ 0.06903 & 1.07326 & 0.02382 \\ 0.62612 & 0.88573 & 1.10105 \end{bmatrix}$$

inverted using MINITAB

$$(I - A)^{-1} D = \begin{bmatrix} 1.06104 & 0.03757 & 0.01222 \\ 0.06903 & 1.07326 & 0.02382 \\ 0.62612 & 0.88573 & 1.10105 \end{bmatrix}\begin{bmatrix} 1000 \\ 1000 \\ 400 \end{bmatrix} = \begin{bmatrix} 1103.49 \\ 1151.81 \\ 1952.26 \end{bmatrix}$$

35. (a) Refer to Section 25.4

(b) Refer to Section 25.4

36. $\mathbf{P}^2 = \begin{bmatrix} 0.2 & 0.8 \\ 0 & 1.0 \end{bmatrix}\begin{bmatrix} 0.2 & 0.8 \\ 0 & 1.0 \end{bmatrix} = \begin{bmatrix} 0.04 & 0.96 \\ 0 & 1.0 \end{bmatrix}$

$\mathbf{P}^4 = \begin{bmatrix} 0.04 & 0.96 \\ 0 & 1.0 \end{bmatrix}\begin{bmatrix} 0.04 & 0.96 \\ 0 & 1.0 \end{bmatrix} = \begin{bmatrix} 0.0016 & 0.9984 \\ 0 & 1.0 \end{bmatrix}$

$\mathbf{P}^8 = \begin{bmatrix} 0.0016 & 0.9984 \\ 0 & 1.0 \end{bmatrix}\begin{bmatrix} 0.0016 & 0.9984 \\ 0 & 1.0 \end{bmatrix} = \begin{bmatrix} 0.00000256 & 0.99999744 \\ 0 & 1.0 \end{bmatrix}$

37. initial vector is [100 50 10 0]

(a) End of Year 1:

$$[100 \quad 50 \quad 10 \quad 0]\begin{bmatrix} 0.7 & 0.2 & 0 & 0.1 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0 & 0.3 & 0.6 & 0.1 \\ 0.1 & 0 & 0 & 0.9 \end{bmatrix} = [85 \quad 48 \quad 11 \quad 16]$$

(b) End of Year 2:

$$[85 \quad 48 \quad 11 \quad 16]\begin{bmatrix} 0.7 & 0.2 & 0 & 0.1 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0 & 0.3 & 0.6 & 0.1 \\ 0.1 & 0 & 0 & 0.9 \end{bmatrix} = [75.5 \quad 44.3 \quad 11.4 \quad 28.8]$$

(c) End of Year 3:

$$[75.5 \quad 44.3 \quad 11.4 \quad 28.8]\begin{bmatrix} 0.7 & 0.2 & 0 & 0.1 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0 & 0.3 & 0.6 & 0.1 \\ 0.1 & 0 & 0 & 0.9 \end{bmatrix} = [69.02 \quad 40.67 \quad 11.27 \quad 39.04]$$

(d) End of Year 4:

$$[69.02 \quad 40.67 \quad 11.27 \quad 39.04]\begin{bmatrix} 0.7 & 0.2 & 0 & 0.1 \\ 0.3 & 0.5 & 0.1 & 0.1 \\ 0 & 0.3 & 0.6 & 0.1 \\ 0.1 & 0 & 0 & 0.9 \end{bmatrix} = [64.419 \quad 37.52 \quad 10.829 \quad 47.232]$$

38.

(a) after one year:

$$[200 \quad 150 \quad 150 \quad 100 \quad 50 \quad 0]\begin{bmatrix} 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0.3 & 0.2 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$= [60 \quad 105 \quad 110 \quad 145 \quad 160 \quad 70]$$

(b) after 4 years:

$$[200 \quad 150 \quad 150 \quad 100 \quad 50 \quad 0]\begin{bmatrix} 0.3 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0 & 0.3 & 0.2 & 0.2 & 0.2 & 0.1 \\ 0 & 0 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0.5 & 0.4 & 0.1 \\ 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \end{bmatrix}^4$$

$$= [1.62 \quad 7.695 \quad 20.63 \quad 66.245 \quad 265.83 \quad 287.98]$$

(c) Total wage bill is multiplication of 2 vectors (leave out last column)

$$[200 \quad 150 \quad 150 \quad 100 \quad 50]\begin{bmatrix} 50 \\ 60 \\ 80 \\ 100 \\ 120 \end{bmatrix} = [47000]$$

(d) after 4 years:

$$[1.62 \quad 7.695 \quad 20.63 \quad 66.245 \quad 265.83]\begin{bmatrix} 50 \\ 60 \\ 80 \\ 100 \\ 120 \end{bmatrix} = [40717.20]$$

39. (a) after 1 period:

$$[100 \quad 5 \quad 0]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0 & 0 & 1.0 \end{bmatrix} = [74.5 \quad 20 \quad 10.5]$$

(b) after 2 periods:

$$[74.5 \quad 20 \quad 10.5]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0 & 0 & 1.0 \end{bmatrix} = [70.15 \quad 14.9 \quad 19.95]$$

(c) after 3 periods:

$$[70.15 \quad 14.9 \quad 19.95]\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.9 & 0 & 0.1 \\ 0 & 0 & 1.0 \end{bmatrix} = [62.515 \quad 14.03 \quad 28.455]$$

(d) after 4 periods (reduced to 2×2 matrix):

$$[62.515 \quad 14.03]\begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix} = [54.9845 \quad 21.5605]$$

(e) after 5 periods (reduced to 2×2 matrix):

$$[54.9845 \quad 21.5605]\begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix} = [55.73755 \quad 20.80745]$$

(f) after 6 periods (reduced to 2×2 matrix):

$$[55.73755 \quad 20.80745]\begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix} = [55.662245 \quad 20.882755]$$

Chapter 26

1. 0
2. 2
3. 3
4. $10x^4$
5. $42x^2 - 12x^3$
6. $34x + 14$
7. $7 + 4y - 12y^2$
8. $4q - 4q-2$
9. e^x
10. $2 + 8s + 0.5s^{-1/2}$
11. $f'(x) = 140x^6 + 24x^5 - 15x^4 - 28x^3 + 12x^2 - 22x + 2 + x^{-2} - 18x^{-3}$
12. $-2x^{-3} + 6x^{-4} + (16/3)x^{-5}$
13. $\text{MR} = 100 - 4x$; $P = 50$
14. $\text{AC} = (1.3)x^2 - 5x + 30$
    $\text{MC} = x^2 - 10x + 30$
15. (a) $x = 19.673\,99$
    (b) $x = 11.888$
16. $\text{MC} = \text{MR}$ at $x = 3.765$
    $\text{AC} = \text{AR}$ at $x = 6.437$
17. (a) $E_D = -1.5$
    (b) $E_D = -1.0$
18. (a) $P = 1000 - 5Q$
    (b) $P = 1500 - 55Q$
    (c) $\text{MR} = 1000 - 10Q$, $\text{MR} = 1500 - 110Q$
    (e) $E_D = -19$, $E_D = -1.727$
    (f) $400 < \text{MC} < 900$
19. Min at $x = 5$, $y = 0$
20. Max at $x = 10$, $y = 1200$
21. Min at $x = -1$, $y = 2$
22. Linear, so no max or min
23. Min at $x = 1.6667$, $y = -4.3333$
24. Max at $x = 1$, $y = 12.33$
    Min at $x = 5$, $y = 1.67$
25. Max at $b = 0.25$, $a = 10.25$
26. Max at $x = 0.5858$, $y = 13.3137$
    Min at $x = 3.414$, $y = -9.3137$
27. Neither, a constant cannot have max or min

28. Min at $x = 1$, $y = 37.75$
Max at $x = 2$, $y = 38$
Min at $x = 3$, $y = 37.76$

29. Output $(x) = 10$: Profit $= 210$

30. Max profit at $x = 11.888$ when profit $= £696.1116$

31. Output $(x) = 3.765$: Profit $= 92.58$

32. Min AC at $x = 1$, when AC $=$ MC $= 98$

33. $f'(x) = 6x^2 + 10x + 12$

34. $(2x^2 + 6x + 5)(6x + 4) + (3x^2 + 4x + 10)(4x + 6)$

35. $f'(x) = 3(4x^3 + 6x^2)^2\,(12x^2 + 12x)$
$= 576x^8 + 2304x^7 + 302x^6 + 1296x^5$

36. $xe^{x^2/2}$

37. $f'(x) = [(x^3 + 10)(4 + 12x) - (4x + 6x^2)(3x^2 + 6)]/[(x^3 + 6x)^2]$
$= [-6x^4 - 8x^3 + 36x^2]/[x^6 + 12x^4 + 36x^2]$

38. $[20 - 18x^4]/(9x^8 + 60x^4 + 100)$

39. $2x^2 + c$

40. $x^3 + 2x^2 + 10x + c$

41. $0.5x^4 + 2x^3 - 10x^{-1} + c$

42. $25x + c$

43. $\dfrac{x^3}{3} + \dfrac{x^2}{2} + 5x + c$

44. 26

45. 200

46. 1433.33

47. (a) $x = 20.53$
(b) 108.93
(c) $E_D = -2.65$
(e) Production level decreases

48. (a) TC $= 16Q - Q^2 + 8$
(b) TR $= 40Q - 8Q^2$
(c) $Q = 2.5$
(d) $Q = 1.71429$
(e) 3 or 0.428 57

49. $\delta y/\delta x = 4z$, $\delta y/\delta z = 4x$

50. $\delta y/\delta x = 6x^2 + 8xz + 22$, $\delta y/\delta z = 4x^2 + 4xz - 9z^2$

51. $\delta y/\delta x = 7 + 6xz - 3x^2z^2 - 2z + 5z^2$,
$\delta y/\delta z = 3x^2 - 2x^3z - 2x + 10xz - 21xz^2$

52. $\delta y/\delta x = 40x^3$, $\delta y/\delta z = -45z^2$

53. $\delta q/\delta p_1 = 4p_1 - 3 + 4p_2$, $\delta q/\delta p_2 = 4p_1 - 5 + 2p_2$

54. $\delta r/\delta t = 2s^2 + 2s + 10 - 12ts + 12s = 2s^2 + 14s - 12ts + 10$,
$\delta r/\delta s = 4ts + 2t - 6t^2 + 12t - 21 = -6t^2 + 14t + 14ts - 21$

55. $z = 3, x = 2, y = 1.5$

56. $z = 5, x = 1, y = 7$

57. $z = 10, x = 5, y = 400$

58. $y = 1.5, x = 2, z = 31$

59. $z = 0, x = 0, y = 0$

60. Linear so no max or min

61. $L = 2, K = 3, \text{TC} = 1900$

62. $X = 2, Y = 2.2, \Pi = 52.6$

63. (a) $Y = 1, X = 5, \Pi = 130$
(b) $Y = 0, X = 5.2, \Pi = 125.2$
(c) $X = 0, Y = 2, \Pi = 10$

64. $z = 0.868, x = 2.605$

65. $z = 0.101\ 77, x = 0.508\ 85$

66. $y = 539.0435, x = 741.4348, z = -716234.7$

67. $y = 2.233, x = 10.2913, z = -91.652\ 71$

68. $y = 0.5, x = 0.5, z = 0.5$

69. $L = 33.333\ 333, K = 20, Q = 103.2796$

70. $K = 119.403, L = 44.776, Q = 747.5016$

71. (a) $x = 1, y = 5, \Pi = 718$
(b) $y = 4.629\ 29, x = 0.185185, \Pi = 715.9252$